Wax

30308

ARPA Network Information Center
Stanford Research Institute
Menlo Park, California 94025

L. Kleinrock
F. Tobagi
UCLA
March 1974

Packet Radio Temporary Note #99
NIC # 30308

Carrier Sense Multiple Access with a Busy Tone (BTMA)

This note treats the Carrier Sense Multiple Access mode with a Busy-Tone also called Busy Tone Multiple Access (BTMA). It corresponds to section 7 of the "Permanent" note on Carrier Sense Multiple Access and is written as such. However it is self-contained and can be considered separately. To facilitate the distribution , it is sent as a temporary note. Some conclusions, pertaining to the optimum choice of some system parameters, are based on G/S, the average number a packet is transmitted and scheduled until success. Packet delays are presently being considered by means of simulation. Results will soon be available.

RECEIVED AT NIC 4-2-74

7. CSMA WITH A BUSY TONE (BTMA)

7.1 Introduction

The busy Tone Multiple Access mode (BTMA) is a natural extension of the Carrier Sense Multiple Access; it is introduced to eliminate the "hidden terminal" problem. Indeed, we recognized in Section 6 that terminals may not all be within range and in line-of-sight of each other as required for the proper operation of CSMA. We had also shown that the degrading effect hidden terminals have on throughput performance can be important.

The operation of BTMA rests on the fact that the station is, by definition, within range and in line-of-sight of all terminals. We assume that the total available bandwidth is divided into two channels: a message channel and a busy-tone (BT) channel. As long as the station is sensing a carrier on the message channel, it transmits a busy-tone signal on the busy-tone channel. It is by sensing the busy tone channel that terminals learn about the state of the channel. The various protocols studied in the previous sections can be used. However, we restrict ourselves to the non-Persistent protocol because of its simplicity in analysis and in implementation, as well as its relatively high efficiency.

In CSMA, the difficulty of detecting the presence of a signal on the message channel when this message used the entire bandwidth was minor and therefore was neglected. It is not so when we are concerned with the (statistical) detection of the busy tone signal on a narrow band channel. The detection time is no longer negligible and must be accounted for; the analysis of the system as well as the optimum choice of the various design variables become more complex. We consider these questions below.

7.2 The Non-persistent BTMA Protocol

The protocol of the non-persistent BTMA is identical to the protocol of the non-persistent carrier sense (see section 2).

Whenever a packet is ready for transmission, the terminal senses (listens to) the busy tone channel for t_d seconds at the end of which either it decides that the BT signal is present or it decides that it is absent. In the first case it reschedules the packet for transmission some time later. In the second case, it transmits the packet after the t_d seconds of sensing.

7.3 Signal Detection

The detection of the busy-tone signal is the problem of detecting a signal of known form in the presence of noise. The useful signal is a given function with some unknown parameters, namely, phase and amplitude.* However, the observation time is usually small compared to the "fluctuation time" of these parameters, and the unknown phase and amplitude can be regarded as constant.

The problem of detecting a signal in a background of random noise is a statistical problem involving the choice of one or two mutually exclusive hypotheses. This has been extensively studied in the past [W1]. The quality of the decision can be characterized by two probabilities:

D probability of correct detection (in presence of the signal)

F probability of incorrect detection or false alarm

Let t_d be the observation time; i.e., the width of the window over which the channel is observed in making the decision. If μ_{t_d} is the signal-to-noise ratio when the signal is present over the entire window t_d, and r_* is the optimum threshold of the statistical detector, then for the usually assumed Rayleigh channel, it can be shown that [W1]

$$F = e^{-r_*^2/2\mu_{t_d}}$$

$$r_* = \sqrt{-2\mu_{t_d} \text{ LogF}} \qquad (7.1)$$

*Because of the mobility of terminals, the signal fluctuates. Thus we assume it to be of unknown amplitude. In the case of fixed terminals, the problem will consist of detecting a signal with known amplitude but unknown phase.

That is, for a given observation time t_d and a desired false alarm probability F, the optimum threshold r_* can be determined from Eq.(7.1).

In this case (i.e., when the useful signal is present over the entire window), the probability of proper detection is given by [W1].

$$D_{t_d} = \exp\left(-\frac{r_*^2}{2\mu_{t_d}(1+\mu_{t_d})}\right)$$

$$= F^{1/(1+\mu_{t_d})} \qquad (7.2)$$

Eq. (7.2) rests on the fact that the signal-to-noise ratio of the received useful signal during the observation time is actually μ_{t_d}. In the following paragraph we investigate the (transient) situations where the signal may be present during only a fraction of the observation window.

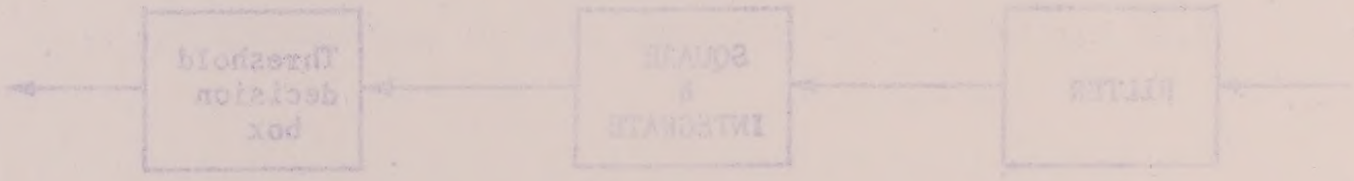
Transient Behavior

Fig. 7.1. Block Diagram of the Busy Tone Signal Detector.

The detector at the receiver consists mainly of a filter, an integrator and the threshold decision box. Assume the step response of the busy tone detect filter is exponential [F1]; the amplitude at time t for the output of the busy tone filter is*

$$A_{BT}(t) = A_{max}\left(1 - e^{\frac{t}{k}}\right) \qquad (7.3)$$

*At time $t = 0$, the filter is just turned on, and the signal is assumed to be present.


FILTER
SQUARE & INTEGRATE
Threshold decision box

where A_{max} is the maximum amplitude and k is the time constant. If we assume that the same peak power is used for the busy tone as for the message on the message channel, then

$$A_{max} = A_m$$

where A_m is the amplitude of the message signal on the message channel. Since the energy of a signal is the integral of its squared amplitude (which equals $\frac{A^2 t}{2}$ for a sinusoid of amplitude A and duration t), we define the signal-to-noise ratio (SNR) as the ratio of the signal energy to the noise-energy and express it as

$$\mu = \frac{A^2}{2N_o W}$$

where W is the bandwidth of the channel under consideration and N_o is the (white) noise power density.

Let μ_m be the SNR of the message on the message channel required for suitable operation (typically $\mu_m = 10$). Then from the last equation, it is clear that

$$A_{max} = A_m = \sqrt{2N_o W_m \mu_m}$$

and

$$A_{BT}(t) = \sqrt{2N_o W_{BT} \mu(t)} = \sqrt{2N_o W_m \mu_m}\left(1 - e^{-t/k}\right)$$

Taking the time constant k to be $1/2W_{BT}$ and denoting by β the fraction of the total bandwidth W assigned to the busy tone channel, we have the following function defining the SNR $\mu(t)$ on the BT channel:

Digitized by the Internet Archive
in 2025 with funding from
Amateur Radio Digital Communications, Grant 151

https://archive.org/details/temporary-note-099

$$\mu(t) = \mu_m \frac{1-\beta}{\beta}\left(1 - e^{-2\beta Wt}\right)^2 \qquad (7.4)$$

Particularly

$$\mu_{t_d} = \mu(t_d) = \mu_m \frac{1-\beta}{\beta}\left(1 - e^{-2\beta Wt_d}\right)^2 \qquad (7.5)$$

As the starting time of a busy tone is unpredictable, the busy tone may not be present during the entire window of t_d seconds. The resulting SNR is a function of the time u over which the busy tone signal is present (see Eq. (7.4)). By the same token, the probability of proper detection is also a function of the time over which the busy-tone signal is present.

Consider now a signal starting at $t = 0$ and terminating at $t = T$. Let $D(t)$ be the probability of correct detection at time t after having observed the channel over t_d seconds (t is the time at which decision is made). $D(t)$ is determined by

$$D(t) = F^{\frac{\mu_{t_d}}{\mu(u)\left[1+\mu(u)\right]}} \qquad (7.6)$$

where

$$u = \begin{cases} t & \text{if } 0 \le t \le t_d \\ t_d & \text{if } t_d \le t \le T \\ T - t + t_d & \text{if } T < t < T + t_d \end{cases}$$

For $t > T + t_d$, the probability of false alarm is F. $D(t)$ is sketched in Fig. 7.2. A detailed graph of $D(t)$ for $0 < t < t_d$ is given in Fig. 7.3 for $\beta = 0.05$ and various values of F and t_d. It is interesting to note that for $0 \le t \le t_d$, $D(t)$ increases quite rapidly, getting close to $D(t_d)$ for $t < t_d$. That is, the signal need not be present over the entire window t_d to achieve a probability of correct detection close to $D(t_d)$.

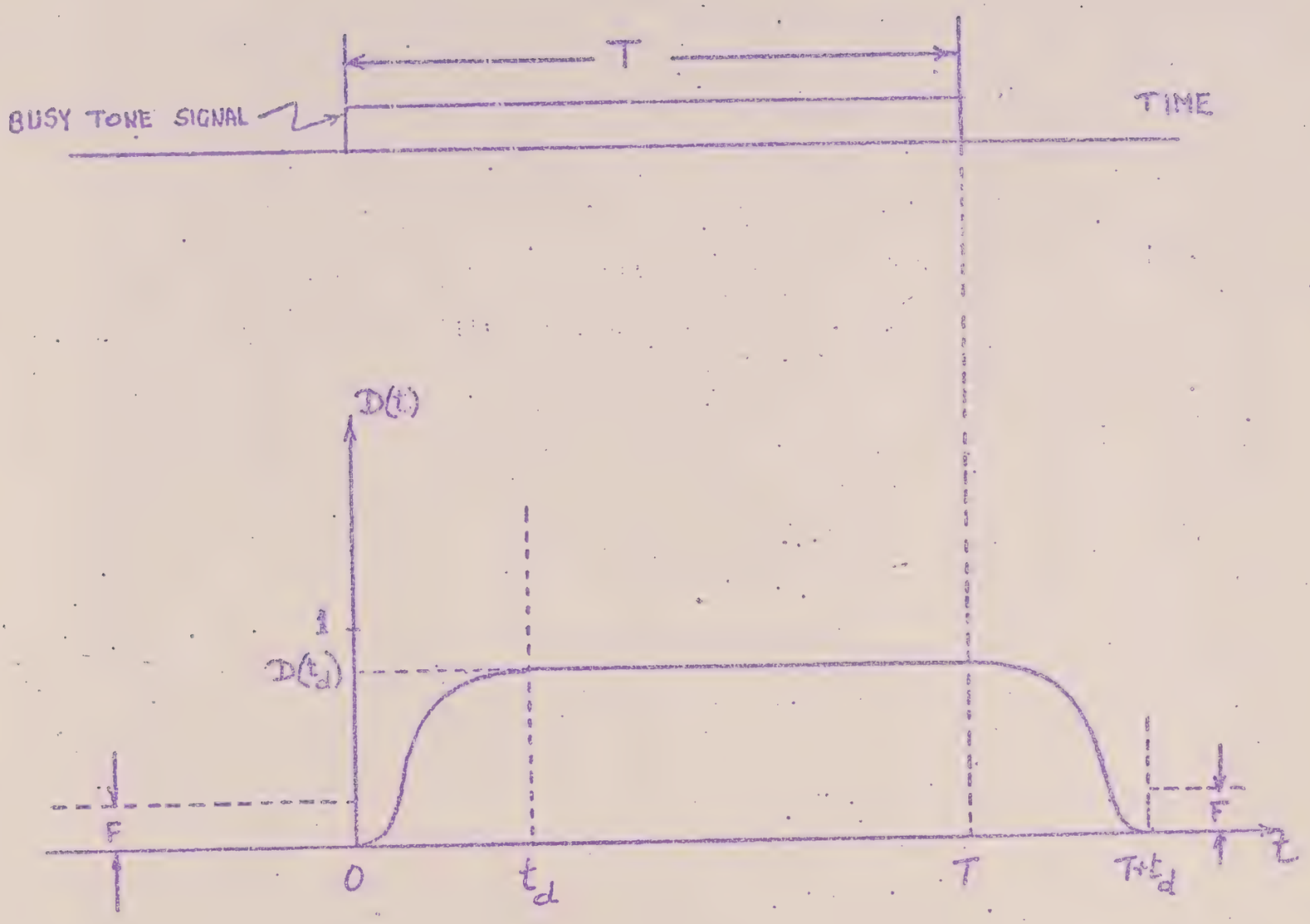
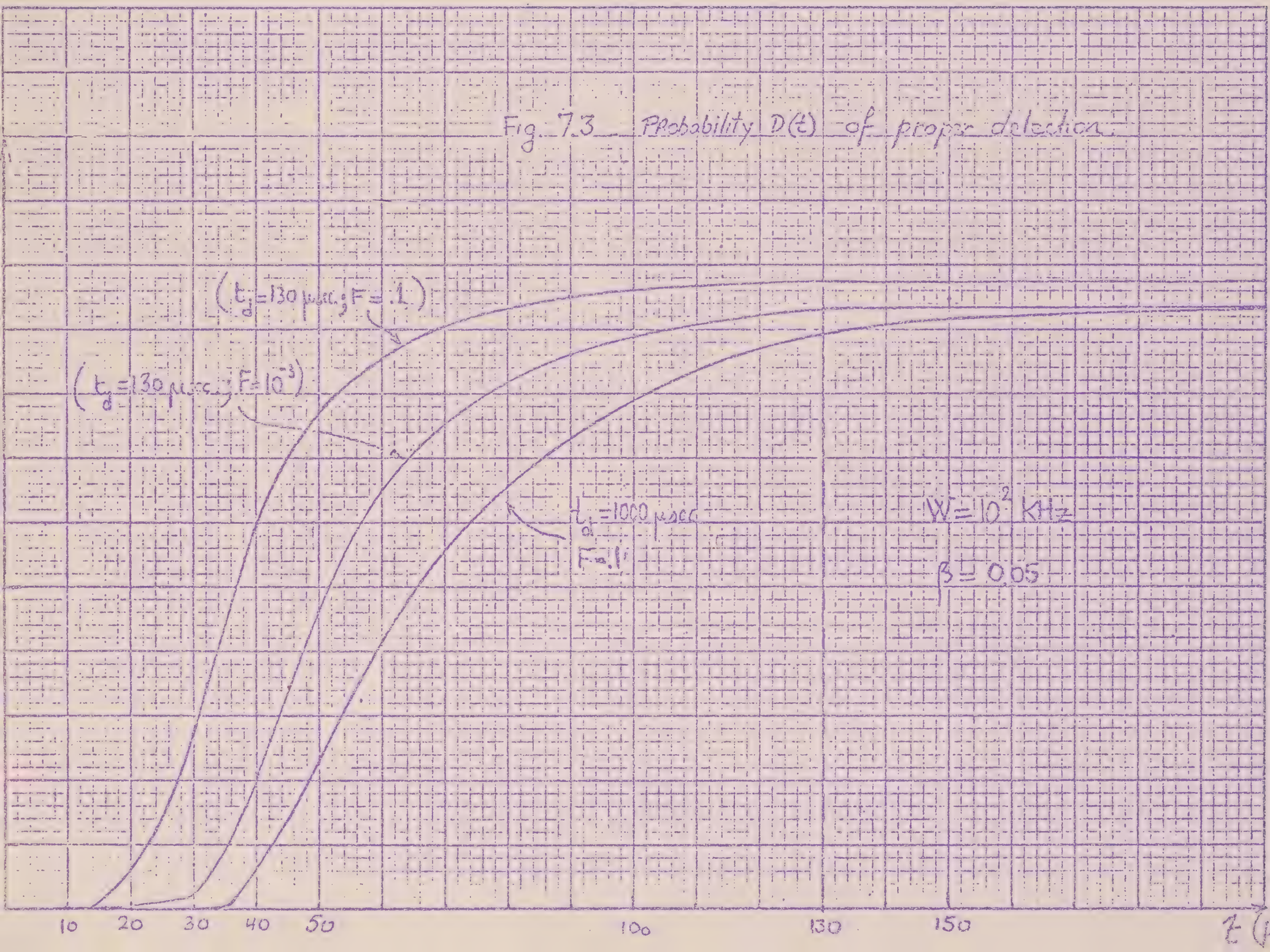


Fig. 7.2. D(t) for an isolated busy tone signal of length T .

KEUFFEL & ESSER CO.

Fig. 7.3 Probability D(t) of proper detection



A more complex situation occurs when two BT signals are separated by a gap shorter than t_d. The window now can overlap over two BT signals.

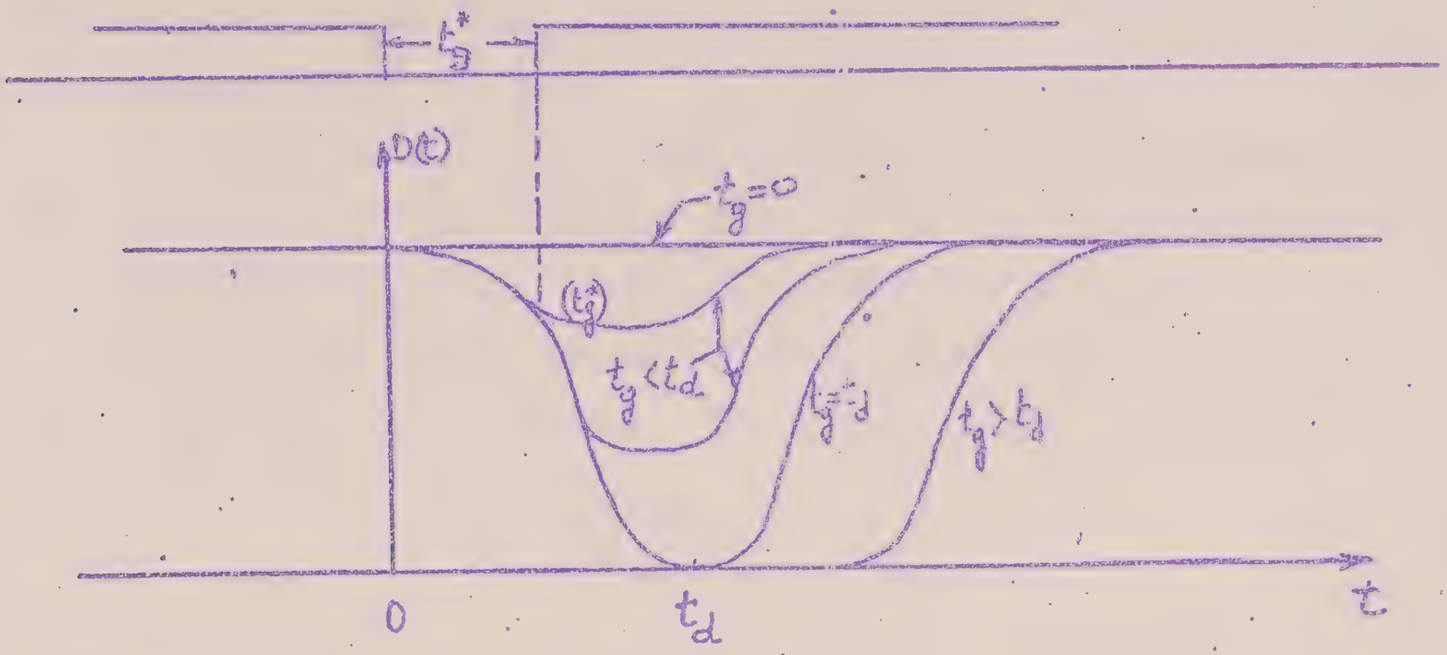


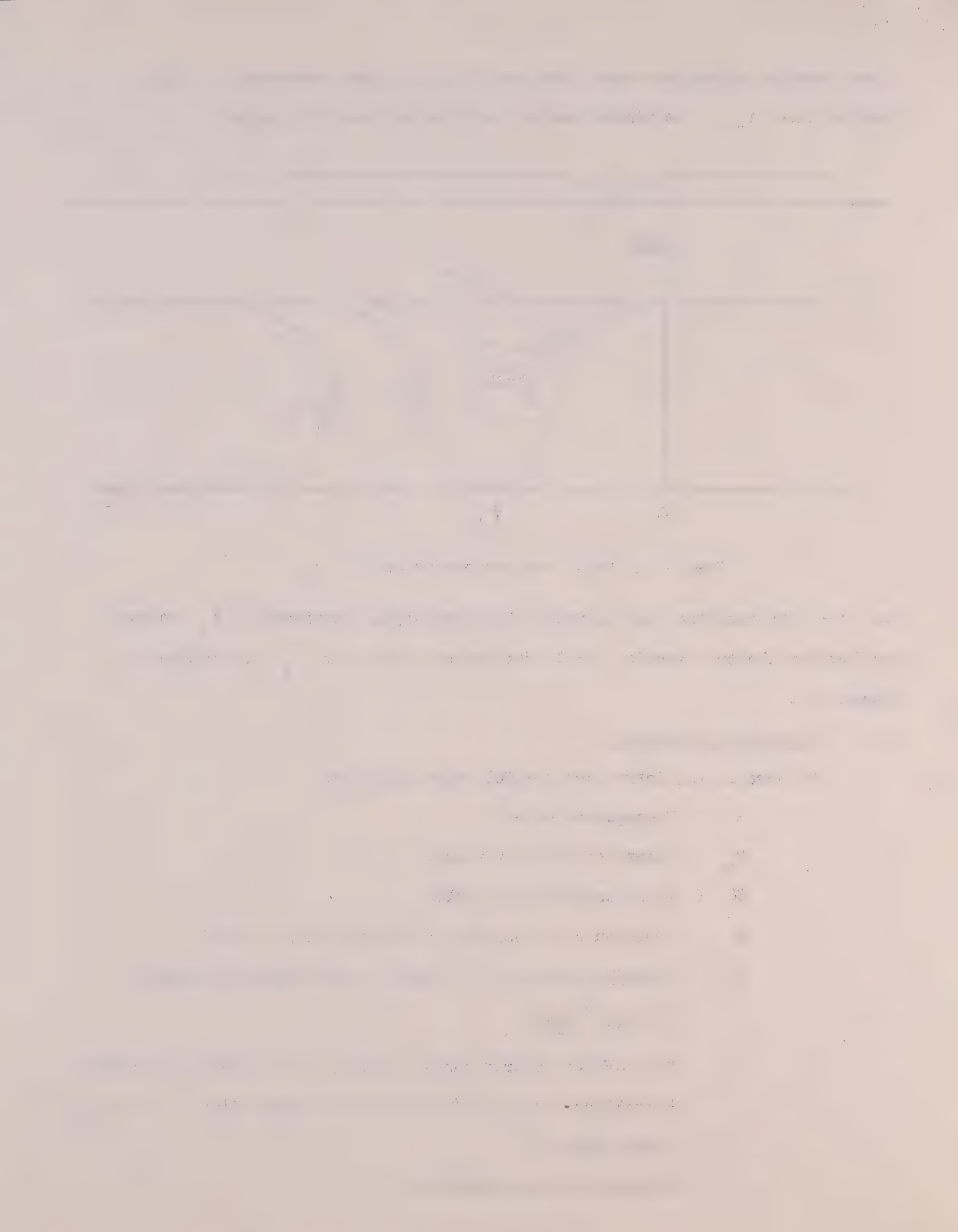
Fig. 7.4. D(t) for various values of t_g.

Let $t = 0$ be here the time at which the first signal terminates, t_g be the gap between the two signals. D(t) for various values of t_g is sketched in Figure 7.4.

7.4 Throughput Equations

We have so far introduced the following notation:

- τ : Propagation delay
- B_m : Number of bits per packet
- W : Total bandwidth available
- β : Fraction of W assigned to the Busy Tone channel
- T_m : Transmission time of a packet on the message channel; $T_m = \frac{B_m}{(1-\beta)W}$
- λ : Rate of the offered channel traffic (in packets per second)
- G : Normalized rate of offered channel traffic (i.e., $G = \lambda T_m$)
- t_d : Detection time
- F : The false alarm probability.



Contrary to the CSMA schemes, the fraction of the population which decides to transmit is a function of time. The analytical approach consists of identifying the busy and idle periods and of determining the condition for success over the busy period as with the CSMA modes.

To keep the analysis simple, some approximations are made yielding to the lower bound on throughput given in the following theorem. Corresponding numerical results are presented and discussed in section 7.5.

Theorem 7.1*: Under stationary conditions and the model assumptions, a lowerbound on the channel utilization, S_ℓ, is given by

$$S_\ell = \frac{E_m}{W} \frac{\exp(-\lambda m(0, T_m))}{\overline{B} + \overline{I}} \qquad (7.7)$$

where

$$m(x_1, x_2) = \int_{x_1}^{x_2} \alpha(t)dt$$

$$\alpha(t) = \begin{cases} 1 - F & 0 \le t + 2\tau \\ 1 - D(t) & 2\tau < t \le 2\tau + t_d \\ 1 - D(t_d + 2\tau) & 2\tau + t_d < t \le T_m \end{cases}$$

$$D(t) = F\left[\frac{\mu(t_d + 2\tau)}{\mu(t)[1 + \mu(t)]}\right]$$

$$\mu(t) = \mu_m \frac{1 - \beta}{\beta}\left(1 - e^{-2\beta W(t - 2\tau)}\right)^2$$

$$\overline{B} = T_m + \frac{1}{\Delta\lambda} e^{\Delta\lambda T_m}\left[1 - Y^*(\Delta\lambda)\right]$$

$$\Delta = 1 - D(t_d + 2\tau)$$

---

*See Appendix E for proof of this theorem.

$$Y^*(s) = e^{-\lambda m(0,T_m)} + \int_0^{T_m} \lambda\alpha(y) e^{-sy} e^{-\lambda m(y,T_m)} dy$$

$$\bar{I} = \int_0^{\infty} e^{-\lambda m'(0,\, z)}\, dz$$

$$m'(z_1,\, z_2) = \int_{z_1}^{z_2} \alpha'(t)dt$$

$$\alpha'(t) = \begin{cases} 1 - D(t_d + 2\tau) = \Delta & 0 \le t \le 2\tau \\ 1 - D'(t) & 2\tau \le t \le 2\tau + t_d \\ 1 - F = 0 & 2\tau + t_d \le t \end{cases}$$

$$D^e(t) = F\left[\frac{\mu(t_d + 2\tau)}{\mu'(t)[1 + \mu'(t)]}\right]$$

$$\mu'(t) = \mu_m \frac{1-\beta}{\beta}\left(1 - e^{-2\beta W(t_d + 2\tau - t)}\right)^2$$

The effect of some of the approximations can be checked by considering another estimate[†] for the channel utilization defined by:

$$S_n = \frac{B_m}{W} \frac{f e^{-\Delta\lambda T_m} + (1-f)\, e^{-m(0,\, T_m)}}{\bar{B}_1 + \bar{I}} \qquad (7.8)$$

where f is the probability that the gap tg is smaller than $t_d + 2\tau$ and is expressed as:

$$f = 1 - e^{-\lambda m'(0,\, t_d + 2\tau)} \qquad (7.9)$$

[†]See Appendix E for the derivations of Eq. (7.8).

and where

$$\bar{B}_1 = T_m + \frac{1}{\Delta\lambda}\, e^{\Delta\lambda T_m}\left[1 - f Y^*_{01}(\Delta\lambda) - (1 - f)\, Y^*(\Delta\lambda)\right]$$

$$Y^*_{01}(s) = \Delta\lambda\, \frac{e^{-sT_m} - e^{-\Delta\lambda T_m}}{\Delta\lambda - s} + e^{-\Delta\lambda T_m}$$

or

$$Y^*_{01}(\Delta\lambda) = 1^{-\Delta\lambda T_m}\,[1 + \Delta\lambda T_m]$$

As discussed in Appendix B, the smaller f is, the closer S_u is to S_ℓ and the more negligible is the effect of these approximations. (For $f = 0$, $S_u = S_\ell$).

7.5 Numerical results and Discussion

The expression given in Eqs. ((7.7), (7.8)) relate the throughput S to the offered channel traffic λ or G. When all the system parameters have fixed values, the information capacity of the channel is defined as the maximum achievable throughput. This throughput is obtained at an optimum value of the traffic G and results in infinite packet delays.

The design problem in BTMA consists of maximizing the channel capacity (under the non-Persistent protocol) by properly selecting the design variables β, F and t_d. B_m, the number of bits per packet, and W, the total available bandwidth, are assumed to be given.

Because of the complicated form of the expressions for S, numerical optimization techniques were used. Below, we first discuss the restrictions on the input data and the accuracy of the approximations. Then we give the numerical results in form of curves. The subsequent curves depict the changes in system performance due to variations in the design parameters. The various trade-offs which influence the performance of BTMA are also discussed.

Restrictions and Approximations

To reduce the dimensionality of the problem, and to provide an easy comparison with the previously analyzed CSMA protocols we restrict ourselves to the following:

- $\mu_m = 10$
- $\frac{Bm}{W} = 10^{-2} = 10$ msec. *

(we considered two cases for B_m and W :

case I : B_m = 1000 bits ; $W = 10^5$ Hz

case II : B_m = 10,000 bits ; $W = 10^6$ Hz

---

*The Bandwidth is assumed to be modulated at 1 bit/Hz.

It is important to note that, for the same μ_m, case II requires higher transmitting power than does case I; the following curves also show that case II offers a channel capacity higher than the one offered by case I.)

- τ (maximum propagation delay) = 100 μsec.

(This corresponds to a maximum distance of about 20 miles. The ratio of propagation delay to transmission time of a packet, denoted by a, is, in all cases less than (but very close to) or equal to 0.01.)

Along with the numerical computation of S_ℓ and S_u we computed the probability f that the idle period is smaller than $t_d + 2\tau$. The probability f never exceeded a few percent and the two estimates on throughput are very close to each other. (As an example, we give in Table 7.1 the values of f encountered for various values of the system parameters). Therefore all numerical results given in the sequel will only correspond to the lower bound S_ℓ.

Results

For $F = 10^{-3}$ and various values of β we plot in Fig. 7.5 the channel capacity versus the observation window t_d. Similar curves can be plotted for other values of F. For each couple (F, β), the channel capacity reaches its maximum at an optimum value of t_d.

This optimum is explained by the fact that the larger t_d is, the better is the probability of correct detection D_{t_d} when the signal is present during the entire window. However, the larger t_d is, the slower is the decrease in $\alpha(t)$ (and hence the level of interference is higher) at the beginning of a busy period. The effect is reversed as t_d gets smaller.

It is interesting to note that when the observation window shrinks to 0, the capacity of the channel decreases to $(1 - \beta)\frac{1}{2e}$ (capacity provided by the pure Aloha access mode). Qualitatively speaking, $t_d \to 0$ reduces to very bad detection,

	λ	S_ℓ	S_u	ε
$\begin{cases} F = 10^{-3} \\ \beta = 10^{-2} \\ t_d = 7.10^{-4} \end{cases}$	10	0.0897	0.0897	0.0019
	100	0.4551	0.4554	0.0189
	400	0.6007	0.6081	0.0737
	500	0.5896	0.6011	0.0913
	600	0.5718	0.5881	0.1085
$\begin{cases} F = 10^{-1} \\ \beta = 2.10^{-2} \\ t_d = 5.10^{-4} \end{cases}$	10	0.0817	0.0817	0.0009
	100	0.4405	0.4406	0.0092
	500	0.6397	0.6439	0.0451
	600	0.6344	0.6406	0.0539
	700	0.6244	0.6327	0.0627
$\begin{cases} F = .5 \\ \beta = 2.10^{-2} \\ t_d = 3.10^{-4} \end{cases}$	10	0.0473	0.0473	0.0005
	100	0.3206	0.3206	0.0052
	500	0.6203	0.6217	0.0259
	900	0.6512	0.6558	0.0462
	1000	0.6493	0.6550	0.0512
$\begin{cases} F = .9 \\ \beta = 5.10^{-2} \\ t_d = 2.10^{-4} \end{cases}$	1000	0.4599	0.4601	0.0128
	2000	0.5776	0.5787	0.0255
	3000	0.6157	0.6183	0.0381
	4000	0.6238	0.6286	0.0504
	5000	0.6179	0.6254	0.0626

TABLE 7.1. Accuracy of the approximations.

K·E SEMI-LOGARITHMIC • 3 CYCLES X 70 DIVISIONS
KEUFFEL & ESSER CO. MADE IN U.S.A.

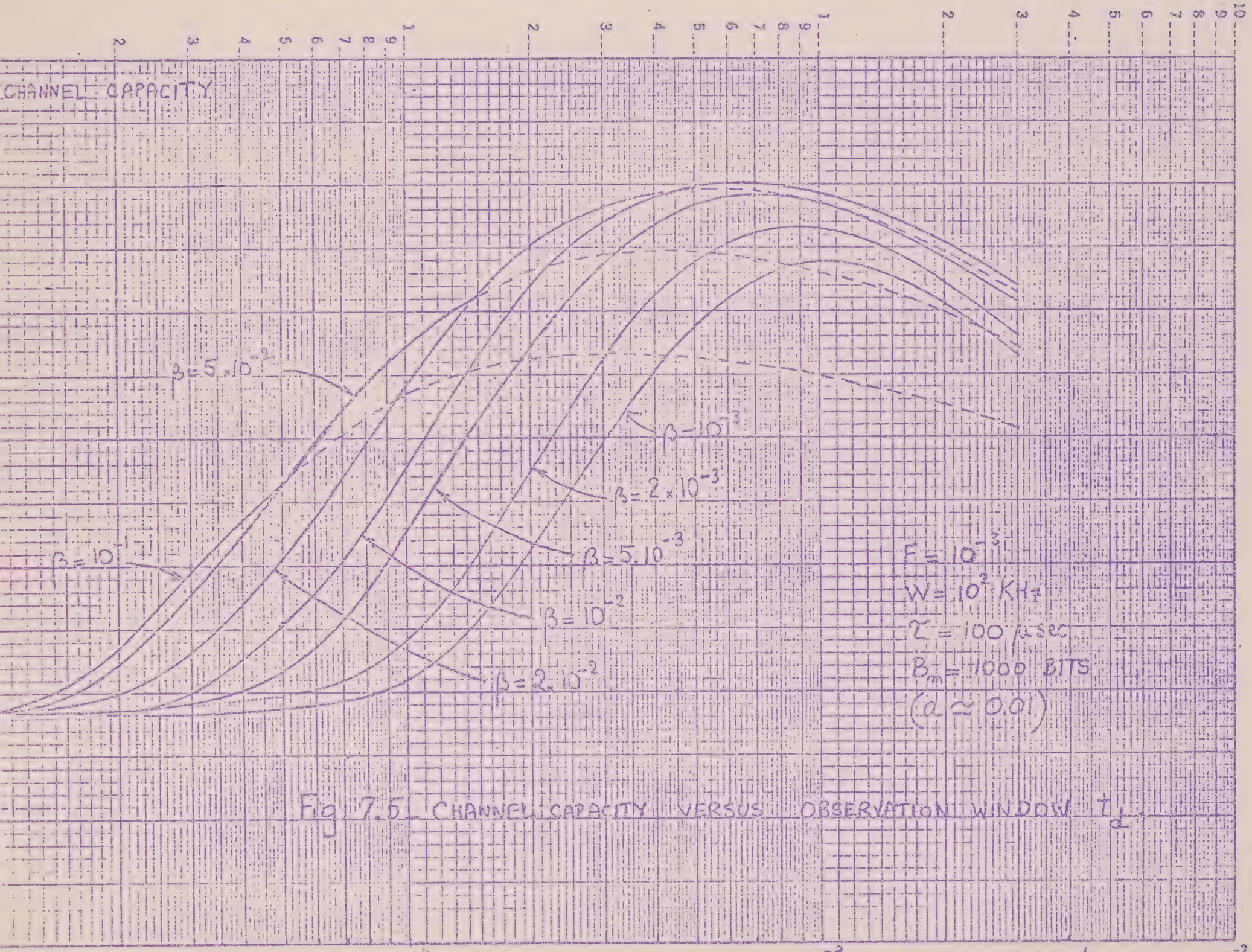


Fig 7.5 CHANNEL CAPACITY VERSUS OBSERVATION WINDOW t_d

and terminals behave in a Pure Aloha mode. In Appendix E, we show that Eq. (7.7), the throughput equation, actually reduces to the pure Aloha throughput equation as $t_d \to 0$.

In Fig. 7.6, we plot the optimum t_d versus β. It is interesting to note that the relationship between β and $(t_d)_{opt}$ is of the form $(t_d)_{opt} = a\beta^b$.

In Fig. 7.7, we plot for each value of F, the maximum capacity of the channel (maximized over t_d) versus β. We note here that the maximum capacity is not very sensitive to small variations of β. That is, for fixed F, BTMA offers good performance over a broad range of β. However there is a certain range of β which yields the best performance. For those values of F considered in the graph ($F = 10^{-3}, 10^{-2}, 10^{-1}, 0.5$), the optimum β is in the range $(10^{-2}, 2 \times 10^{-2})$.

In Fig. 7.8, we plot the capacity (maximized over β and t_d) versus F. Note that for case I and $F < .5$, the capacity of the channel is a logarithmic function of F. The ultimate performance ($= .65$) is obtained for $F = .5$. However, the channel capacity is not very sensitive to variations of F. The effect of F can be explained as follows. For fixed values of β and t_d, the larger F is, the better is the quality of correct detection $D(t)$ (see, for example, Fig. 7.3.). Thus for a particular value of G, the channel time wasted by interference is smaller; the channel idle time, however, which is a function of $(1 - F)G$ is larger; thus a tradeoff exists between idle time and time wasted by interference. The overall performance is not easily expressed. A similar plot, corresponding to $W = 1$ Mhz is also shown. It exhibits the same linear behavior on the semi-logarithmic graph.

To compare the delay performance of BTMA for various values of the system parameters, we consider again the quantity G/S, the average number of transmissions

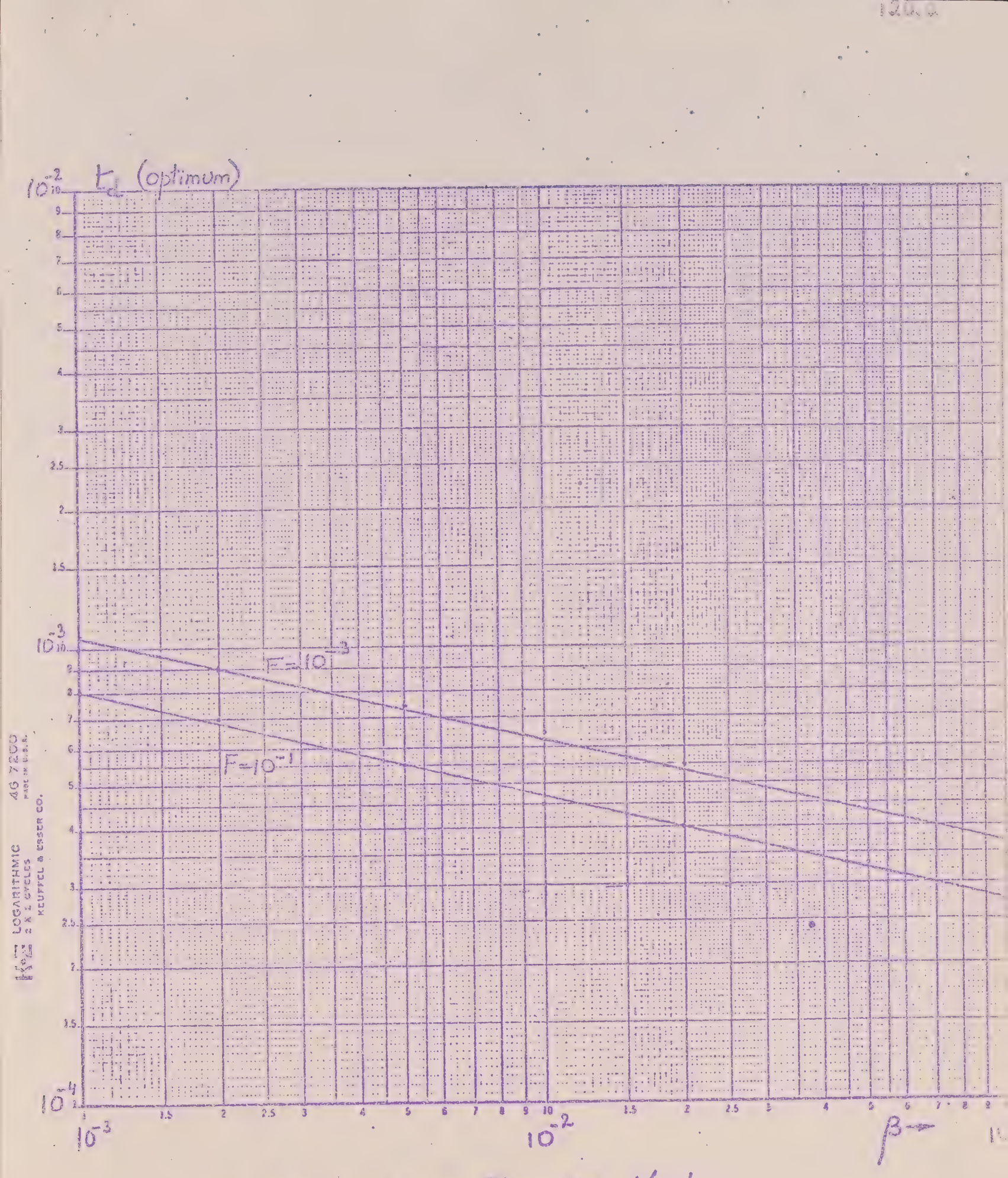


FIG 7.6 - Optimum Observation Window

K&E SEMI-LOGARITHMIC • 3 CYCLES X 70 DIVISIONS
KEUFFEL & ESSER CO. MADE IN U.S.A.

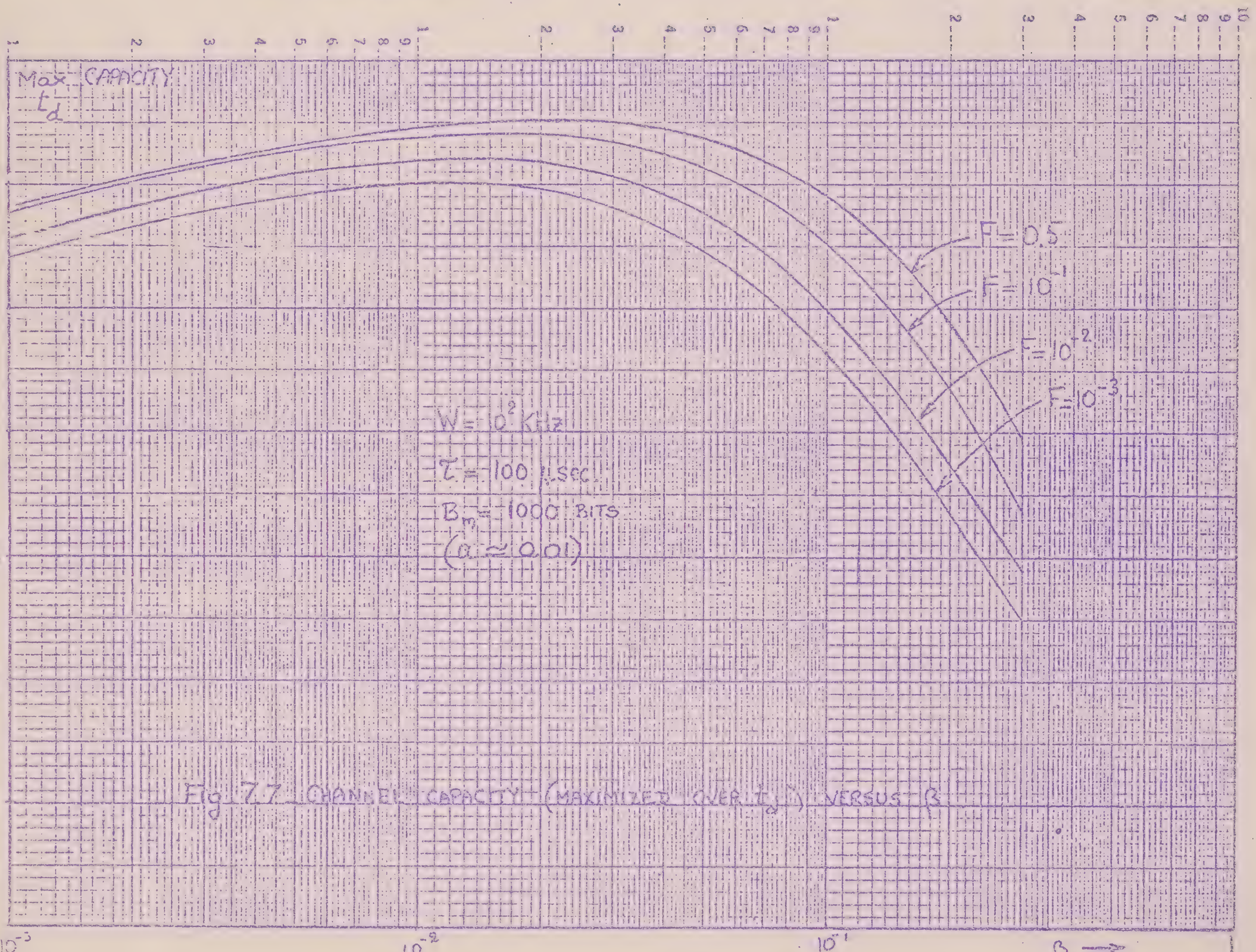


Fig 7.7 CHANNEL CAPACITY (MAXIMIZED OVER t_d) VERSUS β

K+E SEMI-LOGARITHMIC • 3 CYCLES X 70 DIVISIONS
KEUFFEL & ESSER CO. MADE IN U.S.A.

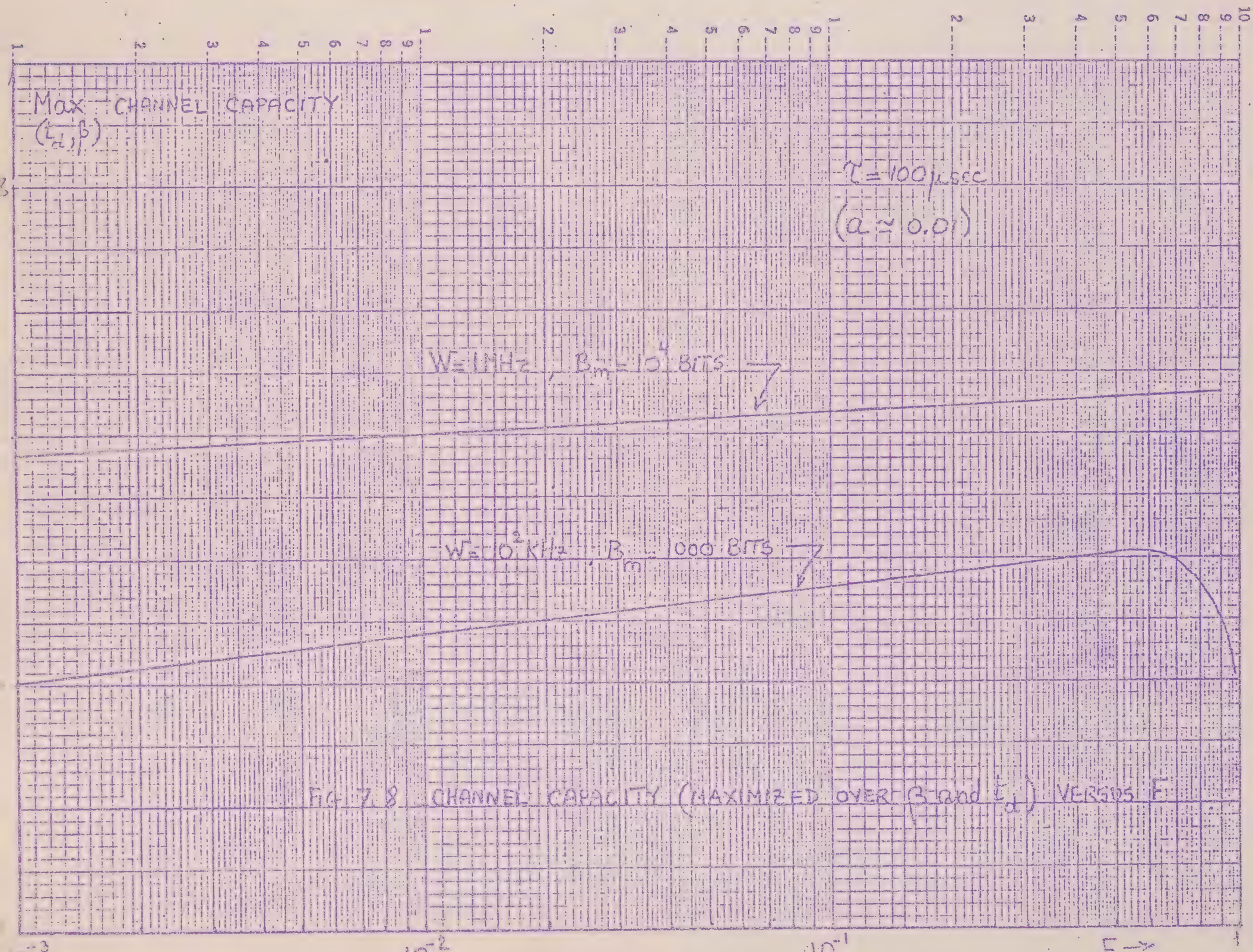


FIG 7.8 CHANNEL CAPACITY (MAXIMIZED OVER β and L_d) VERSUS F

and schedulings that the packet incurs before success.

In Fig. 7.9 we plot, for each value of F, G/S versus S for those values of β and t_d yielding the maximum channel capacity. (Strictly speaking, we should plot G/S versus S for all pairs (β, t_d) and then draw their lower envelope. However, the difference between this lower envelope and the plotted curve corresponding to the optimum β and t_d for maximum capacity is so minor that we restrict ourselves to the latter.)

Note that for each value of S there exists a value of F minimizing G/S. However for relatively small values of S (not too close to the saturation point of the channel) we note that the higher the probability of false alarm F is, the larger is G/S. An a priori explanation can be given by the following fact: when $G \to 0$ and $S \to 0$, the terminal incurs an average number of schedulings equal to $\frac{1}{1-F}$. This is shown on Fig. 7.9 at $S = 0$. (In some cases such as in the curve corresponding to $F = .9$, we have $\lim_{S \to 0} \frac{G}{S}$ slightly larger than $\frac{1}{1-F}$; this is due to the fact that the curve does not correspond to the lower envelope). To best compare the effect F has on delay, we plot in Fig. 7.10 G/S versus F for constant S. We show thus that for $S < 0.5$ and $10^{-3} < F < 10^{-1}$, G/S is low and fairly insensitive to F, and that for $F > 10^{-1}$ it increases rapidly with increasing values of F. For larger values of throughput, the choice of F is more critical. A good operating point should then be in the flat part of the curves.

7.6 Conclusion

The non-Persistent BTMA mode has been analyzed. The approximations made in the analysis are shown to have a very minor effect on channel performance. The channel capacity (optimized over β and t_d) is not very sensitive to F; but when packet delays are considered, we see that (unreasonably) large values of F may have a significant effect even when $S < .5$ and that the choice of F

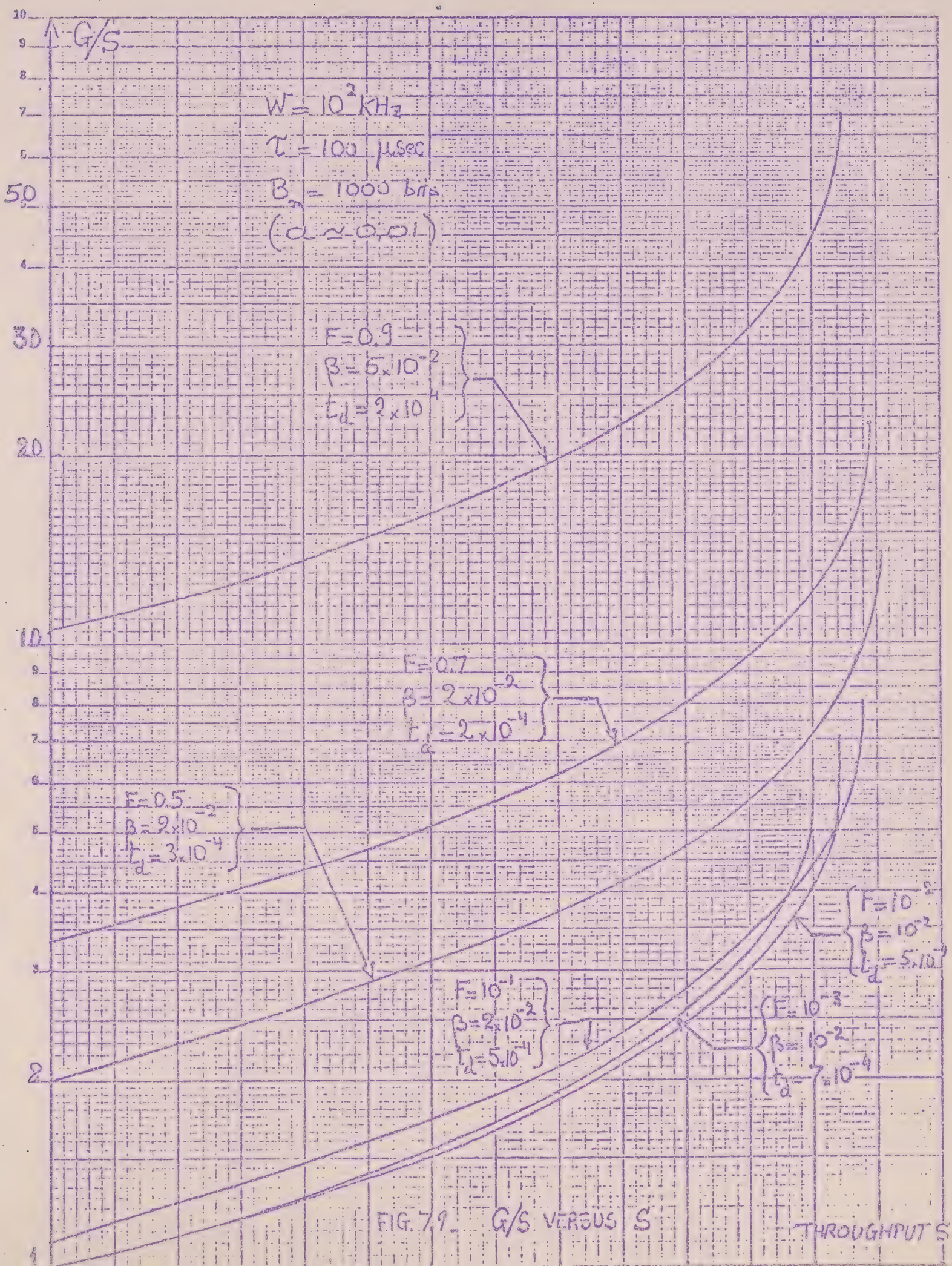


FIG. 7.9 G/S VERSUS S

KEUFFEL & ESSER CO. MADE IN U.S.A. 2 CYCLES X 70 DIVISIONS 46 1510

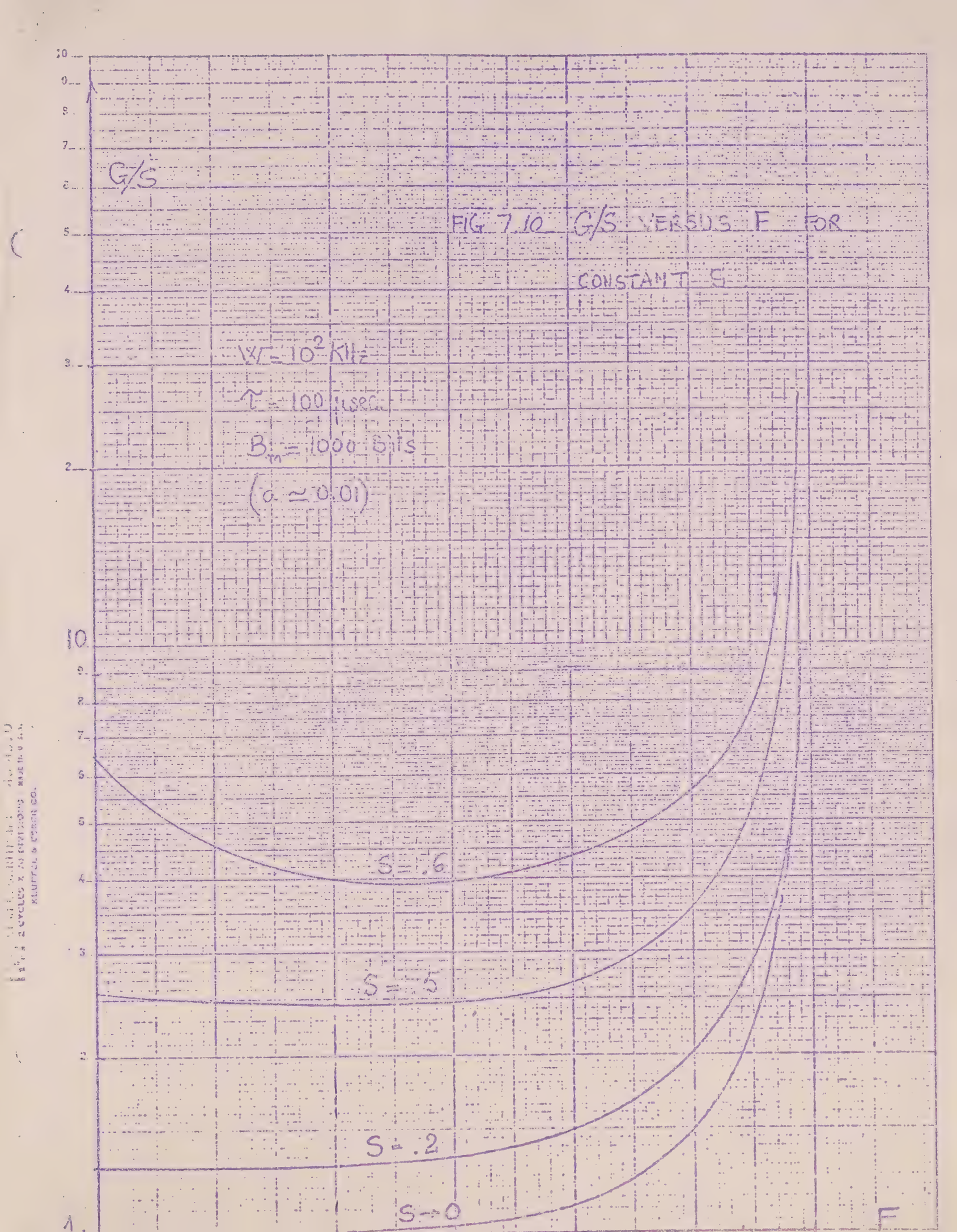


FIG 7.10 G/S VERSUS F FOR CONSTANT S

is more critical for $S > .5$. For case I, and $a \leq 0.01$ to keep the delay low, F should lie in the range $[10^{-3}, 10^{-1}]$. For this range of F the channel capacity lies in the range $[0.60, 0.64]$. These capacities are obtained for the optimum values of β lying in the range $[10^{-2}, 2 \times 10^{-2}]$ and the corresponding optimum values of t_d. Similar results are readily obtainable for other values of W.

Thus the non-Persistent BTMA constitutes a fairly good solution to the "hidden terminal" problem, providing a maximum channel capacity of approximately 0.65 when $a = 0.01$ and $W = 100$ KHz and 0.72 when $a = 0.01$ and $W = 1$ MHz. Furthermore, the performance of the non-Persistent BTMA is insensitive to the precise setting of the various system parameters β, t_d and F.

APPENDIX E

Theorems and Proofs for Section 7

Theorem 7.1 Under stationary conditions and the model assumptions, a lowerbound on the channel utilization, S_ℓ, is given by

$$S_\ell = \frac{B_m}{W} \frac{\exp(-\lambda m(0, T_m))}{\overline{B} + \overline{I}} \qquad (e.1)$$

where the quantities $\overline{B}$, $\overline{I}$, $m(0, T_m)$ are given in Eq. (7.7) and are determined hereafter in the proof.

Proof:

Since the terminal senses the busy tone channel for t_d seconds before deciding to transmit, an arrival is not effective, as far as the channel operation is concerned, until t_d seconds after. To view the occurrence of events in this access mode more easily, we consider the Poisson arrival process to be shifted t_d seconds in time bearing in mind that the terminal has already observed the channel for t_d seconds. We assume as usual all terminals to be at the same distance from the station (see Assumption (A4)); the BT signal emitted by the station is, as seen by the terminals, shifted in time by 2τ, a round trip delay to the station (time until the station hears the transmission) and back (time until the BT from the station is heard by the other terminals).

If λ (packets per second) denotes the offered traffic, the rate of transmission of packets at any time t is $\alpha(t)\lambda$ such that

if the BT signal is present during the entire observation window, then

$$\alpha(t) = 1 - D_{t_d}$$

- if the BT signal is absent during the entire window, then

$$\alpha(t) = 1 - F$$

- and for the case when the BT signal is only present over a fraction of the window, then $\alpha(t)$ can be upperbounded by $1 - D(t)$:

$$\alpha(t) \leq 1 - D(t)$$

where $D(t)$ is the probability of correct detection at time t and F is the false alarm probability. That is, at any time t there is a fraction $\alpha(t)$ of the population which decides to transmit.

Contrary to CSMA, an arrival in the middle of an on-going transmission has a non-zero probability $(1 - D(t))$ of transmitting; thus, a busy period (period of time over which the channel is continuously used) can now exceed $1 + 2\tau$.

Let t_{p1} be the time at which the transmission of the first packet of a busy period starts. The busy tone signal of such a busy period starts 2τ seconds later, at time $t_{p1} + 2\tau$. The time dependent packet transmission rate $\alpha(t)\lambda$ depends on the length of the gap between the start of the busy-tone signal corresponding to the current busy period and the end of the BT signal corresponding to the previous busy period. Let us assume $t_g > t_d + 2\tau$ for the present time. (This corresponds to the worst case). Without loss of generality, let $t_{p1} = 0$. From the previous section, we can easily derive $\alpha(t)$ as follows (see Fig. E.1).

$$\left.\begin{aligned} \alpha(t) &= \begin{cases} 1 - F & 0 < t + 2\tau \\ 1 - D(t)^{\dagger} & 2\tau < t < 2\tau + t_d \\ 1 - D(t_d + 2\tau) & 2\tau + t_d < t \leq T_m . \end{cases} \\ D(t) &= F\left[\frac{\mu(t_d + 2\tau)}{\mu(t)[1 + \mu(t)]}\right] \\ \mu(t) &= \mu_m \frac{1 - \beta}{\beta}\left(1 - e^{-2\beta N(t - 2\tau)}\right)^2 \end{aligned}\right\} \quad (e.2)$$

---

†As mentioned earlier, $1 - D(t)$ is an upper bound on $\alpha(t)$ for $2\tau < t \leq 2\tau + t_d$.

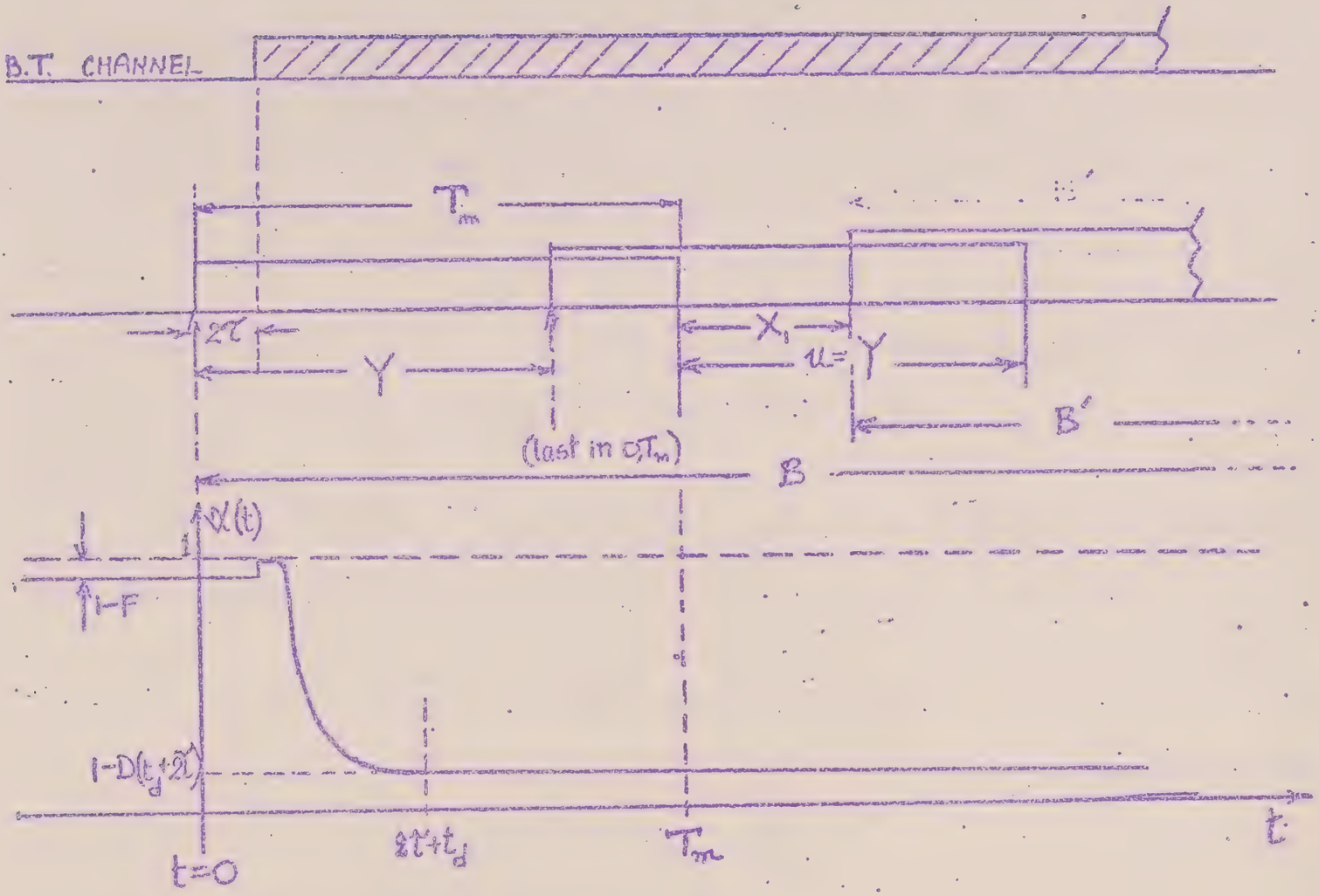


Figure E.1. $\alpha(t)$ at the start of a busy period.

$$\text{Let } m(x_1, x_2) = \int_{x_1}^{x_2} \alpha(t)dt \qquad \text{(e.3)}$$

The first packet in the busy period is successful if there is no arrival, of the non-homogeneous Poisson arrival process with rate $\alpha(t)\lambda$, during the entire transmission time of the packet. The probability of success is given by:

$$\exp(-\lambda m(0, T_m)) \qquad \text{(e.4)}$$

To find the channel throughput, we can use here again the cycle analysis. For this, we need to calculate the average busy and idle periods.

Average busy period

Let Y be the random variable defining the time until last arrival during the packet transmission time T_m (see Fig. E.1). It is easy to see that:

$$\begin{aligned} \Pr\{Y \le y\} &= \Pr\{\text{no arrival in } (y, T_m)\} \\ &= \exp\left(-\lambda m(y, T_m)\right) \qquad \text{(e.5)} \\ \Pr\{Y = 0\} &= \exp\left(-\lambda m(0, T_m)\right) \end{aligned}$$

The Laplace transform $Y^*(s)$ is defined as

$$\begin{aligned} Y^*(s) &= \int_0^{T_m} e^{-sy} d\Pr\{Y \le y\} \\ &= \Pr\{Y = 0\} + \int_0^{T_m} e^{-sy} \lambda e^{-\lambda m(y, T_m)} dm(y, T_m) \\ &= e^{-\lambda m(0, T_m)} + \int_0^{T_m} \lambda\alpha(y) e^{-sy} e^{-\lambda m(y, T_m)} dy \qquad \text{(e.6)} \end{aligned}$$

Let

$$D \equiv D(t_d + 2\tau) \qquad \text{(e.7)}$$

$$\Delta = 1 - D \qquad \text{(e.8)}$$

$$\Phi = 1 - F \qquad \text{(e.9)}$$

The busy period is equal to:

$T_m + Y$ if no arrival occurs in the period of time $(T_m, T_m + Y)$

$T_m + X_1 + B'$ otherwise

where (see Fig. E.1)

- X_1 is the random variable defining the time elapsed from $t = T_m$ up to the first arrival in $(T_m, T_m + Y)$; and
- B' is the length of a busy period when the Poisson arrival process is homogeneous with rate $\Delta\lambda$. (B' is the sub-busy period created by the first arrival in $(T_m, T_m + Y)$. Beyond $t = T_m$, $G(t) = \Delta$) .

Conditioning on $Y = y$, the average busy period is given by

$$\bar{B}_y = (T_m + y)\, e^{-\Delta\lambda y} + (T_m + \bar{X}_{1y} + \bar{B}')(1 - e^{-\Delta\lambda y})$$

$$= T_m + ye^{-\Delta\lambda y} + (\bar{X}_{1y} + \bar{B}')(1 - e^{-\Delta\lambda y})$$

The expected value $\bar{X}_{1y}$ is derived as follows:

$$\Pr\left\{X_1 \le x \;/\; \text{some arrival occurred}, Y = y\right\} = \frac{1 - e^{-\Delta\lambda x}}{1 - e^{-\Delta\lambda y}}$$

$$\bar{X}_{1y} = \frac{1}{1 - e^{-\Delta\lambda y}} \int_0^y (e^{-\Delta\lambda x} - e^{-\Delta\lambda y})\,dx$$

$$= \frac{1}{1 - e^{-\Delta\lambda y}} \left[\frac{1}{\Delta\lambda}(1 - e^{-\Delta\lambda y}) - y\, e^{-\Delta\lambda y}\right] \qquad (e.10)$$

$$= \frac{1}{\Delta\lambda} - \frac{y e^{-\Delta\lambda y}}{1 - e^{-\Delta\lambda y}}$$

The expected busy period $\bar{B}'$ is given by

$$\bar{B}' = \frac{1}{\Delta\lambda}\left[e^{\Delta\lambda T_m} - 1\right] \qquad (e.11)$$

Indeed, we have

$$B' = \begin{cases} T_m & \text{with probability } e^{-\Delta\lambda T_m} \\ X + B' & \text{with probability } 1 - e^{-\Delta\lambda T_m} \end{cases}$$

where X is the time elapsed from the start of transmission of a packet until the start of transmission of the first overlapping packet. $\bar{X}$ is obtained from $\bar{X}_{1y}$ (Eq. (e.9)) by taking $y = T_m$.

$$\bar{X} = \frac{1}{\Delta\lambda} - \frac{T_m e^{-\Delta\lambda T_m}}{1 - e^{-\Delta\lambda T_m}} \qquad (e.12)$$

Therefore

$$\bar{B} = T_m e^{-\Delta\lambda T_m} + (\bar{X} + \bar{B})(1 - e^{-\Delta\lambda T_m})$$

$$= T_m e^{-\Delta\lambda T_m} + \frac{1 - e^{-\Delta\lambda T_m}}{\Delta\lambda} - T_m e^{-\Delta\lambda T_m} + \bar{B}'(1 - e^{-\Delta\lambda T_m})$$

$$= \frac{1}{\Delta\lambda}\left[e^{\Delta\lambda T_m} - 1\right] \qquad \text{QED.}$$

The average busy period $\overline{B}_y$, conditioned on $Y = y$ is given by

$$\overline{B_y} = T_m + ye^{-\Delta\lambda y} + \frac{1 - e^{-\Delta\lambda y}}{\Delta\lambda} - ye^{-\Delta\lambda y} + \frac{1}{\Delta\lambda}\left[e^{\Delta\lambda T_m} - 1\right]\left(1 - e^{-\Delta\lambda T_m}\right)$$

$$= T_m + \frac{1}{\Delta\lambda}\, e^{\Delta\lambda T_m}\left(1 - e^{-\Delta\lambda y}\right) \qquad \text{(e.13)}$$

Removing the condition on Y, we get

$$\overline{B} = \int_0^{T_m} \overline{B}_y \, dP_r\left\{Y \leq y\right\}$$

$$= T_m + \frac{1}{\Delta\lambda} e^{\Delta\lambda T_m}\left[1 - Y^*(\Delta\lambda)\right] \qquad \text{(e.14)}$$

Average idle period

Let $t = 0$ denote now the end of the busy period. It is easy to see here that the transmission rate, which we denote by $\alpha'(t)G$ is defined by (see Fig. E.2)

$$\alpha'(t) = \begin{cases} 1 - D(t_d + 2\tau) = \Delta & 0 \leq t \leq 2\tau \\ 1 - D'(t) & 2\tau \leq t \leq 2\tau + t_d \\ 1 - F = \Phi & t \geq 2\tau + t_d \end{cases} \qquad \text{(e.15)}$$

where

$$D'(t) = F\left[\frac{\mu(t_d + 2\tau)}{\mu'(t)[1 + \mu'(t)]}\right]$$

$\mu(t)$ defined in Eq. (e.2)

$$\mu'(t) = \mu_m \frac{1 - \beta}{\beta}\left(1 - e^{-2\beta W(t_d + 2\tau - t)}\right)^2$$

Denoting by I the idle period, we have

$$Pr\left\{I > z\right\} = Pr\left\{\text{no arrival occurs in } (0,z)\right\}$$

$$= \exp(-Gm'(0,z))$$

where

$$m'(z_1, z_2) = \int_{z_1}^{z_2} \alpha'(t)dt \qquad (e.17)$$

The average busy period is given by

$$\Gamma = \int_0^\infty e^{-\lambda m'(0,\, z)}\, dz \qquad (e.18)$$

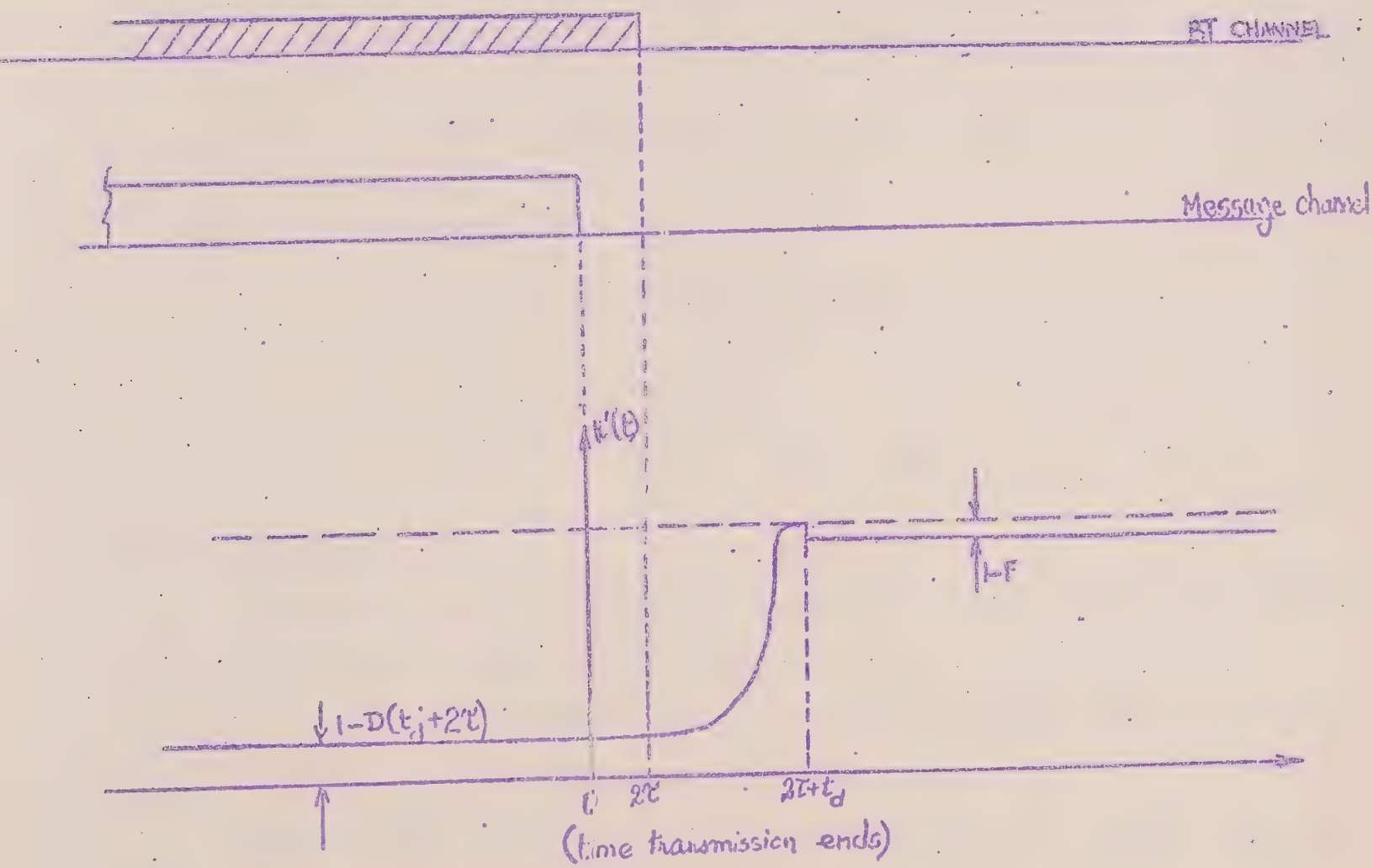


Figure E.2. $\alpha'(t)$ at the start of an idle period.

Channel throughput

The exact expression* for the channel throughput depends on the distribution of the gap t_g. Indeed, $D(t)$ as well as $\alpha(t)\lambda = [1 - D(t)]\lambda$ depends on t_g as shown in Fig. 7.3. However, to keep the analysis fairly tractable, we choose to give a lower bound on channel throughput by considering $\alpha(t)\lambda$ as determined in the worst case ($t_g >> t_d$), that is as given in Eq. (e.2). Under these considerations, the first transmission in the busy period is successful with the probability

$$\exp(-\lambda m(0,T_m))$$

and the expected period of time that the channel is transmitting packets without interference during a cycle (a busy period plus an idle period) is

$$\bar{U} = T_m \exp(-\lambda m(0,T_m))$$

Therefore, the throughput is given by

$$S = (1 - \beta) \frac{T_m \exp(-\lambda m(0,T_m))}{\bar{B} + \bar{I}}$$

$$= \frac{B_m}{W} \frac{\exp(-\lambda m(0,T_m))}{\bar{B} + \bar{I}} \qquad \text{QED.} \qquad \text{(e.1)}$$

"Upper bound" on S

To check the validity of the lower bound provided by Eq. (e.1), we consider the fraction f of busy periods for which the gap t_g is less than $t_d + 2\tau$. Indeed, it is for these cases that $\alpha(t)$ is overestimated by the expression given in Eq. (e.2), and could be underestimated by $1 - D_{t_d} = \Delta$.

The probability that the gap t_g is less than $t_d + 2\tau$ is readily given by

$$f = \Pr\{t_g \leq t_d + 2\tau\} = 1 - e^{-\lambda m'(0,\, t_d + 2\tau)} \qquad \text{(e.20)}$$

---

*Exact under the provision that a is constant (see Assumption (A4)).

The smaller f is, the closer is the lower bound on S to the exact expression. On the other hand, an "upper bound"[†] on S is obtained by underestimating $\alpha(t)$ by Δ for a fraction f of the busy periods. That is, a lower bound on the expected busy period is obtained by

$$\overline{B}_{\ell} = T_m + \frac{1}{\Delta\lambda} e^{\Delta\lambda T_m} \left[1 - f\, Y^*_{01}(\Delta\lambda) - (1 - f)\, Y^*(\Delta\lambda)\right] \quad \text{(e.21)}$$

where $Y^*_{01}(s)$ is the Laplace transform of Y when $\alpha(t) = \Delta\lambda$, and is given by

$$Y^*_{01}(s) = \lambda\Delta \frac{e^{-sT_m} - e^{-\lambda\Delta T_m}}{\lambda\Delta - s} + e^{-\lambda\Delta T_m}$$

so that

$$Y^*_{01}(\Delta\lambda) = e^{-\lambda\Delta T_m}\,[1 + \lambda\Delta] \quad \text{(e.22)}$$

An upper bound on the probability of success of the first transmission in a busy period is obtained by

$$f\, e^{-\lambda\Delta T_m} + (1 - f)\, e^{-\lambda m(0, T_m)}$$

The upper bound on throughput then is provided by

$$S_u = \frac{B_m}{W} \frac{f e^{-\lambda\Delta T_m} + (1 - f) e^{-\lambda m(0, T_m)}}{\overline{B}_{\ell} + \overline{I}} \quad \text{(e.23)}$$

Limit when $t_d \to 0$.

The channel capacity reduces to $(1 - \beta)\frac{1}{2e}$ when $t_d \to 0$.

Proof:

We realize that at the lirit $(t_d \to 0)$, the problems caused by the transient behavior are insignificant (nonexistent when $t_d = 0$). We see then from Eq. (7.2)

[†]With respect to the approximation concerning the gap t_g.

that

$$D_{t_d} = F^{\frac{1}{1+\mu_{t_d}}} \to F$$

since $\mu_{t_d} \to 0$. Therefore $\alpha(t)$ is constant and equal to $\Phi = \Delta = 1 - F$; Eq. (e.6) becomes:

$$Y^*(s) = e^{-\Phi\lambda T_m} + \Phi\lambda\, e^{-\Phi\lambda T_m}\left[\frac{e^{(-s+\Phi\lambda)T_m} - 1}{\Phi\lambda - s}\right]$$

yielding.

$$Y^*(\Delta\lambda) = e^{-\Phi\lambda T_m}\left[1 + \Phi\lambda T_m\right]$$

Therefore $\bar{B}$ as given by Eq. (e.14) becomes

$$\bar{B} = \frac{1}{\Phi\lambda}\, e^{\Phi\lambda T_m} - \frac{1}{\Phi\lambda}.$$

Similarly, we can see that

$$\bar{I} = \frac{1}{\Phi\lambda}$$

Substituting in Eq. (e.1) we get

$$S = (1-\beta)\,\Phi\lambda T_m\, e^{-2\Phi\lambda T_m}$$

yielding a channel capacity equal to $(1-\beta)\,\frac{1}{2e}$.

REFERENCES

[F1] Fralick, S., Private communication. Stanford Research Institute , Menlo Park.

[W1] Wainstein, L.A. and V.D.Zubakov, "Extraction of Signals From Noise", Prentice Hall, 1962.